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EXACT SOLUTIONS OF A NAVIER–STOKES EQUATION IN THE FORM OF POLYNOMIALS

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Exact polynomial solutions of a Navier–Stokes equation that describe flow of a Newtonian incompressible fluid are classified. Four classes of such solutions are recognized: parametric, time, and coordinate solutions and polynomials in inverse powers of the Reynolds number. The procedure of finding the exact binomial solutions is discussed. An example of nonseparating flow about a circle is given.

Keywords: *Newtonian fluid, Navier–Stokes equation, Euler equation, Reynolds number, differential constraints.*

Introduction. The term "exact solution" has no unified definition. We can single out the notion of the "exact solution in closed form," i.e., the solution obtained without numerical methods, when we are able to get rid of integrals and derivatives in the initial differential, integral, or integro-differential equations.

It is expedient to subdivide all possible solutions into determinate and stochastic ones and those determinate into stable and unstable solutions. A stable determinate solution can, in principle, be obtained with numerical methods and be identified with the exact solution in a broad sense. Another approach is in subdividing mathematical models into correct, incorrect, and conventionally correct. The solution of a correct model can also be obtained using numerical methods and be identified with the exact solution in a broad sense.

The notion "exact solution" itself is not universally accepted in science. It has been replaced by the notion "integrability of a system of equations" [1], which intuitively corresponds to the notion of a regular behavior of the solution (of a laminar flow in hydrodynamics). There is no standard algorithm enabling us to obtain all exact solutions of a system of differential equations; moreover, solutions expressed by elementary functions are absent for most problems. Three methods of finding the exact solutions are most commonly use.

1. *Search for the First, Second, and More General Integrals of a System.* A system is assumed to be totally integrable if all its integrals have been found [1].

2. *Symmetry Analysis.* Its sources have been presented in the fundamental works of Sophus Lie and his disciples. It is precisely Lie who was first to propose the method of diminishing the number of independent variables using a group-theoretical analysis [2]. A group-theoretical classification of the exact solutions of a Navier–Stokes equation has been given in [3]. It has been established that the basic group of continuous transformations of this equation is infinite. The Lie algebra and its basis have been found. It has been shown that a Navier–Stokes equation allows a Galilean group, as all classical mathematical models of the Newtonian mechanics do.

3. *Qualitative* (*Topological*) *Integration*, i.e., finding the trajectory field of a system of equations. Such an approach involves a local analysis of the solution in the vicinity of singular elements and numerical solution of the problem beyond these vicinities [4].

The current attitude toward the exact solutions is dual. On the one hand, every *exact* solution describes the *exact* properties of flow. On the other, every exact solution, just as every integrable system of equations, does not describe the turbulent regime of flow.

Although the term "exact solution" has yet to become universally accepted, the solution in the form of a convergent series or polynomial is always assumed to be exact. It satisfies the sought system of equations with all its terms being preserved. It is well known that series of special functions, e.g., hypergeometric ones, are broken, becoming polynomials, with a certain selection of the parameters. Most existing exact solutions of a Navier–Stokes equation are polynomials [5–9]. The first nontrivial exact solutions of ideal-gas equations in the form of polynomials were proposed by Euler [7].

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Essence of the Method. The proposed method is used for finding the exact solutions of a Navier–Stokes equation which, for a Newtonian incompressible fluid, has the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\,\mathbf{u} + \nabla p = \nu\Delta\mathbf{u}\,, \quad \nabla\mathbf{u} = 0\,, \tag{1}$$

where the fluid density is equal to unity.

The solution method involves the expansion of the sought functions in parametric, time, or coordinate polynomials. The terms of the polynomials can be represented in closed form and in the form of a convergent series and be determined from a system of ordinary differential equations or a system of linear partial equations. Since the number of equations necessary for computing the coefficients of such a polynomial and the overdetermination of the problem dramatically grow with the number of its terms, we are led to restrict ourselves to the binomial expansion of velocity. The problem's overdetermination diminishes, since the expansion for pressure is trinomial. The idea of using such expansion, which is based on the fact that pressure appears linearly in (1), has been used, e.g., in [10], for seeking the exact solution in the form of the coordinate power expansion of a polar angle.

Let us elucidate the essence of the method with the example of the expansion

$$\mathbf{u} = \mathbf{u}_1 + \sigma\mathbf{u}_2\,, \quad p = p_1 + \sigma p_2 + \sigma^2 p_3\,. \tag{2}$$

The parameter σ to be determined in the exact solution of (2) plays the role of an arbitrary constant not appearing in $\mathbf{u}_1$, $\mathbf{u}_2$, p_1, p_2, and p_3.

Substituting (2) into Eq. (1), we recognize three groups of terms: those of the order of $O(1)$, $O(\sigma)$, and $O(\sigma^2)$. Setting each of them equal to zero, we obtain three systems of equations: Euler, Navier–Stokes, and constraint equations. Such an operation yields 11 equations for 9 unknowns in the three-dimensional case (determinacy deficit equal to $11 - 9 = 2$) and 8 equations for 7 unknown functions in the two-dimensional case (determinacy deficit equal to $8 - 7 = 1$).

The expansion (2) implies a superposition of viscous and nonviscous flows. The constraint equations determine compatibility conditions under which such a (nonlinear) superposition is possible. The first terms ($\mathbf{u}_1$ and p_1) of the polynomial (2) satisfy unsteady Navier–Stokes equations, and the terms that follow ($\mathbf{u}_2$ and p_3) satisfy steady-state Euler equations. The constraint equation has the form

$$\frac{\partial \mathbf{u}_2}{\partial t} + (\mathbf{u}_1\nabla)\,\mathbf{u}_2 + (\mathbf{u}_2\nabla)\,\mathbf{u}_1 + \nabla p_2 = \nu\Delta\mathbf{u}_2\,. \tag{3}$$

The above three systems of equations cannot be analyzed comprehensively. Therefore, in this work, we consider just two-dimensional particular cases demonstrating the correctness of the method. Using a particular example, we show that a solution exists. Let us consider two plane flows with velocity components u and v:

(1) $u_1(x, y, t) = -x\dfrac{\partial v_1}{\partial y} + u_0(y, t)$, where $u_0(y, t)$ and $v_1(y, t)$ are arbitrary functions of their arguments.

(2) $u_2(x, y, t) = ay$ and $v_2 = p_3 = 0$.

From (3), we find

$$\frac{\partial u_2}{\partial t} + av_1 + ay\frac{\partial u_1}{\partial x} + \frac{\partial p_2}{\partial x} = 0\,, \quad \frac{\partial p_2}{\partial y} = 0\,.$$

Hence we infer that

$$y\frac{\partial v_1}{\partial y} - v_1 = b\,(t)\,, \quad p_2\,(x, y, t) = abx\,.$$

Consequently, we have

$$v_1 = -b(t) + yc(t)\,. \tag{4}$$

The solution ($\mathbf{u}_2$, p_3) satisfies the steady-state Euler equation. It only remains for us to check whether the solution ($\mathbf{u}_1$, p_1) satisfies unsteady Navier–Stokes equations. From (1), we find

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y} + \frac{\partial p_1}{\partial x} = \nu\frac{\partial^2 u_1}{\partial y^2}\,,\quad \frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial y} + \frac{\partial p_1}{\partial y} = \nu\frac{\partial^2 v_1}{\partial y^2}\,. \tag{5}$$

From the second equation of (5), we infer that

$$p_1 = \nu\frac{\partial v_1}{\partial y} - \frac{1}{2}v_1^2 - \int\frac{\partial v_1}{\partial t}\,dy + A(x, t)\,,$$

where the function $A(x,\ t)$ is determined from the first equation of (5) and is equal to $A(x,\ t) = -xg(t) + \frac{1}{2}x^2(c' - c^2)$; the time derivative is primed.

The expansion of (4) satisfies this system on condition that the function u_0 is the solution of a linear equation of the parabolic type with an arbitrary function $g(t)$:

$$\frac{\partial u_0}{\partial t} - \nu\frac{\partial u_0}{\partial y^2} + (b - cy)\frac{\partial u_0}{\partial y} - cu_0 = g(t)\,.$$

The constant a appears in the solution only as the product σa; therefore, we can set $a = 1$. Then the resulting exact solution of the Navier–Stokes equation will take the form

$$u(x, y, t) = -cxy + u_0(y, t) + \sigma y\,,\quad v(x, y, t) = -b + cy\,,$$

$$p(x, y, t) = \nu c - \frac{1}{2}(b - cy)^2 - \frac{1}{2}c'y^2 - b'y - gx + \frac{1}{2}x^2\left(c' - c^2\right) + \sigma bx\,.$$

Parametric Inverse-Power Expansion of the Re Number. Let us consider the series $\mathbf{u} = \mathbf{u}_1 + \sigma(\nu)\mathbf{u}_2$. Substituting it into (1), we find five groups of terms, which are equal in the orders of $O(1)$, $O(\sigma)$, $O(\sigma^2)$, $O(\nu)$, and $O(\nu\sigma)$. Expansion for the pressure p does not change the situation. Consequently,

T h e o r e m 1 holds. *The overdetermination minimum will be at* $\sigma = \nu$, *when the number of the groups is reduced to 3:* $O(1)$, $O(\sigma)$, *and* $O(\sigma^2)$.

Then (2) takes the form

$$\mathbf{u} = \mathbf{u}_1 + \nu\mathbf{u}_2\,,\quad p = p_1 + \nu p_2 + \nu^3 p_3\,. \tag{6}$$

The Re number is in inverse proportion to ν. Unsteady Euler equations hold true for determination of the first terms of the expansion ($\mathbf{u}_1$ and p_1). The steady-state Navier–Stokes equation at $\nu = 1$ holds true for determination of the second terms of the expansion ($\mathbf{u}_2$ and p_3). The constraint equation is linear for both $\mathbf{u}_1$ and $\mathbf{u}_2$:

$$\frac{\partial \mathbf{u}_2}{\partial t} + (\mathbf{u}_1\nabla)\,\mathbf{u}_2 + (\mathbf{u}_2\nabla)\,\mathbf{u}_1 + \nabla p_2 = \nu\Delta\mathbf{u}_1\,. \tag{7}$$

Restricting ourselves again to the particular example of plane flow, we superpose two solutions: 1) $u_1 = a(t)x$ and $V_1 = -a(t)y$ and 2) $u_2 = A(t)y^2 + B(t)y + C(t)$, and $v_2 = 0$. The first solution satisfies the Euler equation; here, we have $2p_1(x,\ y,\ t) = -x^2(a' + a^2) + y^2(a' - a^2)$. The second solution satisfied the steady-state Navier–Stokes equation; here, time acts as the parameter, and $p = 2A\nu x$.

It only remains for us to check whether the solution satisfies the constraint equations (7), which take the form

$$\frac{\partial u_2}{\partial t} + v_1 \frac{\partial u_2}{\partial y} + a u_2 + \frac{\partial p_2}{\partial x} = 0 \,, \quad a u_2 + \frac{\partial p_2}{\partial y} = 0 \,.$$

Hence we find: $p_2(x, y, t) = -ay\left(\frac{1}{3}Ay^2 + \frac{1}{3}By + C\right) - x(C' + aC)$, $A' = aA$, and B = const. Consequently, the exact solution has the form

$$u\,(x, y, t) = a\,(t)\,x + \nu\,\left[A\,(t)\,y^2 + By + C\,(t)\right] \,, \quad v\,(x, y, t) = -\,a\,(t)\,y \,, \quad p\,(x, y, t)$$

$$= -\frac{1}{2}x^2\,(a' + a) + \frac{1}{2}y^2\,(a' - a) - \nu ay\left(\frac{1}{3}Ay^2 + \frac{1}{2}By + C\right) - \nu x\,(C' + aC) + 2\nu^2 xA\,(t) \,.$$

The Re-independent solution will be obtained if we set $\mathbf{u}_2 = p_2 = p_3 = 0$ in (6). Then both sides of the Navier–Stokes equation (1) will be equal to zero. The zero left-hand side is the Euler equation describing nonviscous-fluid flow with an arbitrary vorticity. This arbitrariness is determined precisely from the equality of the right-hand side to zero ($\Delta\mathbf{u} = 0$). From a vector analysis, it is known that $\Delta\mathbf{u} = -[\nabla\omega]$. It follows that $[\nabla\omega] = 0$, i.e., $\omega = \nabla\Omega$, where Ω is the vortex potential. This means that the vorticity has a potential, i.e., a harmonic function. The practical significance of such flow is attributable to the fact that its stability is independent of the Re number.

Let us consider two particular cases. In plane flow, we have $u_z = \omega_x = \omega_y = 0$; therefore, $\Omega = kz$ and ω_z = const. The second case is axisymmetric flow with a twist. In the cylindrical coordinate system r, θ, x, we have

$$\omega_r = -\frac{1}{r}\frac{\partial\Gamma}{\partial x} = \frac{\partial\Omega}{\partial r} \,, \quad \omega_\theta = \frac{\partial u_r}{\partial x} - \frac{\partial u_x}{\partial r} = \frac{1}{r}\frac{\partial\Omega}{\partial x} \,, \quad \omega_x = \frac{1}{r}\frac{\partial\Gamma}{\partial r} = \frac{\partial\Omega}{\partial x} \,,$$

where $\Gamma = ru_\theta$. If we have $\Gamma \equiv 0$, the vorticity ω_θ will be equal to k/r.

We can extend the class of ν-dependent solutions if we assume, instead of (6), that the flow velocity is independent of the parameter ν and the pressure is in direct proportion to ν: $p = p_1 + \nu p_2$. For such solutions, which will be called *semidependent on* Re, two equations of (1) are split into three Re-free equations: $\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} + \nabla p_1 = 0$, $\nabla\mathbf{u} = 0$, and $\nabla p_2 = \Delta\mathbf{u}$. Such a solution states the identity of the unsteady velocity field of nonviscous flow to the quasi-steady-state velocity field of creeping flow when the pressure distribution in these fields is different. Examples of semidependent solutions have been given in [8–13]. The semidependent solution becomes a dependent solution if $p_2 \equiv 0$.

Time Polynomials. We find out first from which time-dependent functions $\sigma(t)$ time polynomials can be constructed. Let $\mathbf{u}(\mathbf{r}, t)$ be equal to $\mathbf{u}_0(\mathbf{r}) + \sigma(t)\mathbf{u}_1(\mathbf{r})$. Then Eq. (1) will contain four groups of terms having the orders of $O(\sigma')$, $O(1)$, $O(\sigma)$, and $O(\sigma^2)$.

T h e o r e m 2 holds. *The overdetermination minimum will occur if the number of these groups is reduced to 3, which is possible in the following cases:* 1) $\sigma' = O(1)$, *i.e.*, $\sigma = t$, 2) $\sigma' = O(\sigma)$, *i.e.*, $\sigma = \exp(kt)$, 3) $\sigma' = O(\sigma^2)$, *i.e.*, $\sigma = 1/t$.

Let us consider plane flow at $\sigma = t$ as an example. The velocity expansion will be represented in the form

$$u\,(x, y, t) = u_0\,(x, y) + tu_1\,(x, y) \,, \quad v\,(x, y, t) = v_0\,(x, y) + tv_1\,(x, y) \,.$$

The Navier–Stokes equation will be taken in "vortex–stream function" variables:

$$\frac{\partial\omega}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} = \nu\Delta^2\psi \,, \quad \omega = -\,\Delta\psi \,. \tag{8}$$

Expansions for the vorticity and the stream function are also binomial:

$$\omega(x, y, t) = \omega_0(x, y) + t\omega_1(x, y), \quad \psi(x, y, t) = \psi_0(x, y) + t\psi_1(x, y).$$

The functions ω_1 and ψ_1 are determined from the steady-state Euler equation, whereas the functions ω_0 and ψ_0 are determined from the steady-state Navier–Stokes equation with an extra term

$$\omega_1 + \frac{\partial \psi_0}{\partial y}\frac{\partial \omega_0}{\partial x} - \frac{\partial \psi_0}{\partial x}\frac{\partial \omega_0}{\partial y} = \nu\Delta\omega_0, \quad \omega_0 = -\Delta\psi_0. \tag{9}$$

The constraint equation has the form

$$\frac{\partial \psi_0}{\partial y}\frac{\partial \omega_1}{\partial x} + \frac{\partial \psi_1}{\partial y}\frac{\partial \omega_0}{\partial x} - \frac{\partial \psi_0}{\partial y}\frac{\partial \omega_1}{\partial x} - \frac{\partial \psi_1}{\partial x}\frac{\partial \omega_0}{\partial y} = \nu\Delta\omega_1, \quad \omega_1 = -\Delta\psi_1. \tag{10}$$

Let us consider a particular solution: $\omega_0 = 0$. Then we find from (9) that $\omega_1 = 0$. i.e., the solution is vortex-free, on the whole. Both terms appearing in the expansion for the stream function are determined from the Laplace equation: $\Delta\psi_0 = \Delta\psi_1 = 0$. It only remains for us to determine the pressure. It is representable in the form of a polynomial: $p(x, y, t) = p_0(x/y) + tp_1(x/y) + t^2 p_2(x/y)$. The terms p_2, u_1, and v_1 are determined from the steady-state Euler equation. The term p_0 is found from the equations determining u_0 and v_0:

$$u_1 + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} + \frac{\partial p_0}{\partial x} = 0, \quad v_1 + u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y} + \frac{\partial p_0}{\partial y} = 0.$$

After simple computations, we obtain the sought equation for determining p_0: $\frac{1}{2}\Delta p_0 = \left(\frac{\partial u_0}{\partial y}\right)^2 - \left(\frac{\partial u_0}{\partial x}\right)^2$.

The term p_1 is determined from the constraint equations:

$$u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_1}{\partial y} + v_1\frac{\partial u_0}{\partial y} + \frac{\partial p_1}{\partial y} = 0, \quad u_0\frac{\partial v_1}{\partial x} + u_1\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_1}{\partial y} + v_1\frac{\partial v_0}{\partial y} + \frac{\partial p_1}{\partial y} = 0.$$

Hence we easily obtain the sought equation for determining p_1:

$$\frac{1}{2}\Delta p_1 = \frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x} - \frac{\partial u_0}{\partial y}\frac{\partial u_1}{\partial y}.$$

Thus, the solution of the Navier–Stokes equation that is independent of the Reynolds number and infinitely growing with time has been obtained as a result of a certain superposition of two vortex-free flows of an ideal fluid. The pressure is equal to

$$p = 2\Delta^{-1}\left[\left(\frac{\partial u_0}{\partial y}\right)^2 - \left(\frac{\partial u_0}{\partial x}\right)^2\right] + 4t\Delta^{-1}\left(\frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x} - \frac{\partial u_0}{\partial y}\frac{\partial u_1}{\partial y}\right) + t^2\left[\text{const} - \frac{1}{2}\left(u_1^2 + v_1^2\right)\right].$$

Coordinate Polynomials. Let us consider the Navier–Stokes equation (1) in Cartesian coordinates x, y, z. Linearization of the coordinates (one, two, or three) implies a reduction in the dimensions of the problem of 1, 2, or 3. Accordingly, three possibilities appear.

1. The solution that is linear in all coordinates has the form

$$u_x = xu_1 + yu_2 + zu_3, \quad u_y = xv_1 + yv_2 + zv_3, \quad u_z = xw_1 + yw_2 + zw_3,$$

where nine unknown coefficients u_i, v_i, and w_i (i = 1, 2, and 3) are dependent only on the time t. Therefore, the equation of vorticity transfer $\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u}\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\nabla)\mathbf{u} - \nu\nabla^2\boldsymbol{\omega} = 0$ is substantially simplified: two terms, the second and the fourth, disappear. Finally, we have

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - (\boldsymbol{\omega}\nabla)\,\mathbf{u} = 0\,. \tag{11}$$

The problem is underdetermined: with account for the continuity equation for finding nine time-dependent unknowns (u_i, v_i, w_i), expression (11) yields a system of four ordinary differential equations. Therefore, five functions can be prescribed arbitrarily.

2. Seeking the solution that is linear in one coordinate generally leads to an awkward and overdetermined system of equations. Therefore, we restrict ourselves to the demonstration of a particular case.

In the Cartesian coordinate system x, y, z with a separated x axis, there is the symmetric (in the coordinates y and z) solution

$$u_x = xu_1\,(y, z, t) + u\,(y, z, t)\,, \quad u_y = v\,(y, z, t)\,, \quad u_z = w\,(y, z, t)\,, \quad p = \frac{1}{2}x^2 p_1\,(t) + xp_2\,(t) + p_0\,(y, z, t)\,.$$

To determine five unknown functions (u_1, u, v, w, p_0) of two variables, expression (1) yields a system of five equations. For low u_1 values, such flow is a superposition of the arbitrary flow in the y, z plane and the longitudinal flow along the x axis, which is induced by the pressure gradient p_2.

Similarly we construct the solution in cylindrical coordinates x, r, θ:

$$u_x = u_0\,(r, \theta, t) + xu_1\,(r, t)\,, \quad u_r = v_0\,(r, \theta, t)\,, \quad u_\theta = w_0\,(r, \theta, t)\,, \quad p = p_0\,(r, \theta, t) + xp_1\,(r, \theta, t) + \frac{1}{2}x^2 p_2\,(r, \theta, t)\,.$$

Such solutions yielded by the transition $3D \to 2D$ are used in modeling flows in the shear layer and are a generalization of the solutions at the stagnation point of both the laminar (Himentz solution and Homann solution) and turbulent flows [14]. Analogous solutions are constructed in gas dynamics [5].

Equation (1) describing, in the cylindrical coordinate system x, r, θ, stationary axisymmetric flow of a viscous fluid with a twist contains the exact solution

$$u_x = u\,(r) + xu_1\,(r)\,, \quad u_r = v\,(r)\,, \quad u_\theta = w\,(r)\,, \quad p = p_0\,(r) + ax + \frac{1}{2}bx^2\,.$$

Equations determining u_0, p_0, and w are solved irrespective of the equations determining u_1 and v. The latter have the form

$$v' + \frac{1}{r}v + u_1 = 0\,, \quad vu_1' + u_1^2 + b = \nu\left(u_1'' + \frac{1}{r}u_1'\right).$$

In the case of the coordinate polynomial the question arises of whether its continuation to a series is possible. Let there be necessary to continue the exact polynomial solution $\mathbf{u} = \sum_{n=1}^{N} \sigma^{n-1}\mathbf{u}_n$ to the series $\mathbf{u} = \sum_{n=1}^{\infty} \sigma^{n-1}\mathbf{u}_n$. Substitution of the series into (1) yields a linear homogeneous equation for determination of the $(N+1)$th term $\mathbf{u}_{N+1}$. In the actual problem where there are initial and/or boundary conditions they will also become linear and homogeneous. Therefore, the term $\mathbf{u}_{N+1}$ will be determined accurate to an arbitrary constant C: the polynomial is unambiguously not continual.

If the initial equations are hyperbolic-type, we can set $C = 0$: no continuation is required. An example is provided Prandtl–Mayer flow. If the initial equations are elliptic, knowledge of the solution of the total problem throughout its existence domain is required for analytical continuation of the polynomial in indices N. An example is provided by the already mentioned flow in the vicinity of the critical point.

3. The solution that is linear in two coordinates (x and y) has the form

$$u_x = xu_1\,(z, t) + yu_2\,(z, t) + u_0\,(z, t)\,, \quad u_y = xv_1\,(z, t) + yv_2\,(z, t) + v_0\,(z, t)\,.$$

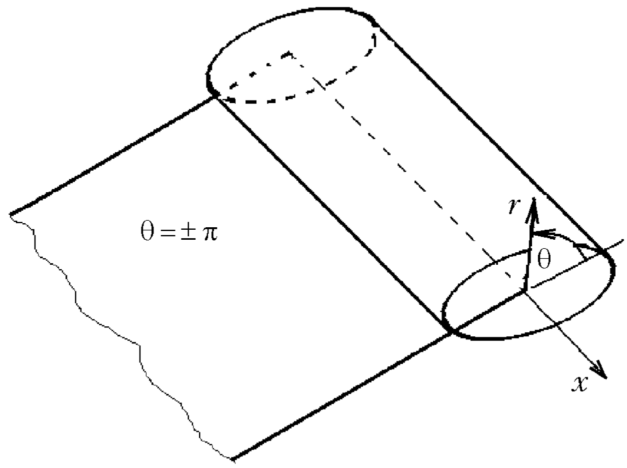

Fig. 1. Body in flow with a velocity-discontinuity plane.

The term $u_z \frac{d\mathbf{u}}{dz}$ is responsible for the nonlinearities in (1), i.e., for terms of the order of $O(x^2, y^2, xy)$. These terms are equal to zero if u_z is independent of x and y: $u_z = w(z, t)$. Then the pressure is represented in the form

$$p = \frac{1}{2} x^2 p_1(t) + \frac{1}{2} y^2 p_2(t) + xyp_3(t) + xp_4(t) + yp_5(t) + p_0(z, t) .$$

For eight sought functions (u_i, v_i, w, and p_6, where i = 0, 1, and 2) of two variables z and t, expression (1) yields a system of eight equations: the problem is determined.

Let us consider the solution that is linear in θ:

$$u_r = u(x, r, t) , \quad u_\theta = v_0(x, r, t) + \theta v(x, r, t) , \quad u_x = u_x(x, r, t) ,$$
$$p(x, r, t) = p_0(x, r, t) + \theta A(x, r, t) + \theta_2 B(x, r, t) . \tag{12}$$

It describes flow past an axisymmetric body $r = r_0(x, t)$ or flow inside it (see Fig. 1) if the nonflow condition is fulfilled:

$$\left[1 + \left(\frac{\partial r_0}{\partial x}\right)^2\right] \frac{\partial r_0}{\partial t} = u_r + u_x \frac{\partial r_0}{\partial x} .$$

The plane $\theta = \pm\pi$ is a surface source of strength $2\pi v(x, r, t)$. Such a solution provides an example of nonseparating flow past a bluff body. This has turned out to be possible, since the incident flow is inhomogeneous. When $r_0 = 0$ the body in flow shrinks to a singular line. Also, there can be flow between two axisymmetric bodies. From (1), we find that u_r and u_θ are independent of x. Therefore, representation (12) can be rewritten in the form

$$u_r = u(r, t) , \quad u_\theta = v_0(r, t) + \theta v(r, t) , \quad u_x = u_0(r, t) + xu_1(r, t) ,$$
$$p = p_0(r, t) + c_1 x + c_2 x^2 + \theta A(r, t) + \theta_2 B(r, t) . \tag{13}$$

This is the transition from the solution that is linear in one coordinate (x) to a solution that is linear in two coordinates (x and θ). Substituting (13) into (1), we obtain four successively solvable systems of equations one of which serves to determine u_1, u_2, v, and B, the second is used to determine v_0 and A, the third serves to determine p_0, and the fourth serves to determine u_0.

The first system will be represented as

$$\frac{\partial u_1}{\partial t} + u\frac{\partial u_1}{\partial r} + u_1^2 + 2c_2 = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_1}{\partial r}\right), \quad v^2 = r\frac{\partial B}{\partial r}, \tag{14}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}(u+v) + \frac{2}{r}B = \frac{\nu}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{v}{r}\right], \quad \frac{\partial (ru)}{\partial r} + v + ru_1 = 0\,.$$

The second system is linear with respect to v_0 and A:

$$2vv_0 = r\frac{\partial A}{\partial r}, \quad \frac{\partial v_0}{\partial t} + u\frac{\partial v_0}{\partial r} + \frac{v_0}{r}(u+v) + \frac{1}{r}A = \frac{\nu}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial v_0}{\partial r}\right) - \frac{v_0}{r}\right].$$

The third system consists of one equation:

$$p_0 = \int\left\{\frac{\nu}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{1}{r}u - \frac{2}{r}v\right] - \frac{\partial u}{\partial t} - u\frac{\partial u}{\partial r} + \frac{1}{r}v_0^2\right\}dr\,.$$

The fourth system is the equation linear in u_0

$$\frac{\partial u_0}{\partial t} + u\frac{\partial u_0}{\partial r} + u_0u_1 + c_1 = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_0}{\partial r}\right).$$

When $c_2 = 0$ a solution of the type of a stationary source

$$u = \frac{a}{r}, \quad v = \frac{b}{r}, \quad u_1 = \frac{c}{r^2} \tag{15}$$

is existent. From (15), we find $a = \frac{1}{2}c - 2\nu$, $b = -c$, and $B = -\frac{1}{2}\frac{c^2}{r^2}$.

Conclusions. Polynomial solutions are widely used in mathematical physics. The efficiency of their use in hydrodynamics for seeking the exact solutions of a Navier–Stokes equation has been shown. The general idea of the method, which diminishes the overdetermination of the problem, is the expansion of pressure in a trinomial series, not the representation of it in the form of a binomial that is used to describe velocity.

The condition of minimum overdetermination of the problem, which is used for selection of the binomial velocity expansion and for determination of the dependence $\sigma(\nu)$ and $\sigma(t)$, is essentially the principle of maximum simplicity (Occam's "razor"). Also, it has been used in selecting the simplest physical model of a medium: an incompressible Newtonian fluid.

There is no successive algorithm for obtaining the exact solutions of a Navier–Stokes equation. The method proposed in this work should be considered as a technique for establishing which of the existing exact solutions can form a new exact solution in polynomial form. The method answers the question the superposition of which of the two solutions of nonlinear equations is possible. Such an approach is applicable not only to a Navier–Stokes equation but also to other nonlinear equations of mathematical physics (see, e.g., [15]). The problem of compilation of today's reference book of exact solutions of not only Navier–Stokes equations but boundary-layer equations as well remains topical [16, 17].

The exact solutions of the equations, which are determined formally, disregarding the initial and boundary conditions, can correspond to the problem of flow if we are able to "place" a solid body in a prescribed velocity field. This is always possible in the case of ideal-fluid flow: any stream surface that turns out to be open can be taken as the solid-body surface. In the case of viscous-fluid flow we can select, as the body's surface, a surface the tangential velocity to which is equal to zero. Generally speaking, we will have the injection/suction of the fluid on such a surface.

The proposed method is a particular case of the more general *method of additional constraints* where certain a priori conditions are set for the form of the sought functions. In this manner, G. I. Taylor, Campe de Ferrer, G.

Gamel, E. Beltramie, I. S. Gromeka, and many other authors have obtained the solution in closed form (see [7]). If the additional constraints are differential, the method is called the *method of differential constraints* [18].

Polynomial solutions can be considered as solutions with a generalized separation of variables [19].

NOTATION

a, b, c, c_1, and c_2, constants; p, pressure; p_k (k = 0, ..., 6), pressure-expansion terms; Re, Reynolds number; t, time; $\mathbf{u}$, velocity vector; $\mathbf{u}_1$ and $\mathbf{u}_2$, velocity-vector-expansion terms; u_x, u_y, and u_z, velocity-vector components along the Cartesian coordinate axes x, y, and z; ψ, stream function; ν, coefficient of kinematic viscosity; $\boldsymbol{\omega}$, vorticity vector; ω, vorticity in plane flow; ω_x and ω_y, vortex-vector components along the Cartesian axes x and y; ω_r and ω_θ, vortex-vector components along the axes of the cylindrical coordinate system x, r, and θ; ∇, gradient operator; Δ, Laplacian.

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